

# A Characterization of Radial Graphs

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## Abstract

A level graph  $G = (V, E, \lambda)$  is a graph with a mapping  $\lambda : V \rightarrow \{1, \dots, k\}$ ,  $k \geq 1$ , that partitions the vertex set  $V$  as  $V = V_1 \cup \dots \cup V_k$ ,  $V_j = \lambda^{-1}(j)$ ,  $V_i \cap V_j = \emptyset$  for  $i \neq j$ , such that  $\lambda(v) = \lambda(u) + 1$  for each edge  $(u, v) \in E$ . Thus a level planar graph can be drawn with the vertices of every  $V_j$ ,  $1 \leq j \leq k$ , placed on a horizontal line, representing the level  $l_j$ , and without crossings of edges, which can be drawn as straight line segments between the levels. Healy, Kuusik and Leipert gave a complete characterization of minimal forbidden subgraphs for level planar graphs (MLNP patterns) for hierarchies [4]. Minimal in terms of deleting an arbitrary edge leads to level planarity. A *radial graph* partitions the vertex set on radii, which can be pictured as concentric circles, instead of levels,  $l_j = (j \cos(\alpha), j \sin(\alpha))$ ,  $\alpha \in [0, 2\pi)$ , mapped around a shared center, where  $j$ ,  $1 \leq j \leq k$  indicates the concentric circles' radius. Comparing embeddings of radial graphs with that of level graphs we gain a further possibility to place an edge and eventually avoid edge crossings which we wish to prevent for planarity reasons. This offers a new set of minimal radial non planar subgraphs (MRNP patterns). Some of the MLNP patterns can be adopted as MRNP patterns while some turn out to be radial planar. But based on the radial planar MLNP patterns and the use of augmentation we can build additional MRNP patterns that did not occur in the level case. Furthermore we point out a new upper bound for the number of edges of radial planar graphs. It depends on the subgraphs induced between two radii. Because of the MRNP patterns these subgraphs can either consist of a forest or a cycle with several branches. Applying the bound we are able to characterize extremal radial planar graphs.

**Keywords:** radial graphs, minimal non-planarity, extremal radial planar graphs

# 1 Introduction

In the context of social network analysis it has become more and more interesting to graphically visualize the information collected by such a social network. Links in hierarchies, in kinship or in financial exchanges for instance must be displayed easily to comprehend for the reader. This is where level graphs come in. Their vertex set is partitioned and there are no edges connecting two vertices within the same partition. The vertices of each partition lie on horizontal lines, the levels. They are suited to display hierarchical network structures. According to Purchase [7] the number of edge crossings has the most important effect on human understanding. The less the number of crossings the more aesthetically the drawn graph appears. Therefore a graph without any crossing, a level planar graph, is wished for. The problem of determining a drawing with a minimal number of crossing is a difficult problem. At the same time a test for level planarity can be done efficiently, see [5]. In [4] Healy et al. were able to characterize level planar graphs by giving a complete list of minimal level non planar patterns, minimal in terms of deleting an arbitrary edge leads to level planarity. The two leveled  $K_{2,2}$  is an example of such a non level planar embeddable graph.

Hence we are interested in a generalization of level graphs, the so called radial graphs. In a radial drawing the vertex partition are no longer drawn on horizontal lines but on concentric circles called radii. Utilizing this generalization we are eventually able to avoid crossings which cannot be prevented in a level graph. Applying it to the level non planar graph  $K_{2,2}$  we are able to draw an edge around the radian and receive a radial planar embedding which can be pictured as a  $C_4$ . In order to test for radial planarity efficiently, we are interested in the structure and properties of radial graphs. Kuratowski's theorem gives a forbidden graph characterization of planar graphs while Healy et al. came up with minimal forbidden patterns of hierarchical level planar graphs. Following this approach we are able to characterize the hierarchical radial planar graphs with regard to minimal radial non planar patterns in Section 3. Another statement about planar graphs can be derived from the Eulerian Formula. That is to say the number of edges in a planar graph is at most  $3n - 6$ . Adapted to radial planar graphs we achieve the bound  $2n - 4$ . Making use of our characterization of forbidden patterns we can classify the induced subgraphs between each two radii. With that in mind we are able to present a new upper bound for the number of edges in a radial graph. This finally leads to a characterization of extremal radial planar graphs which can be found in Section 4. We summarize our work and open problems in the last section.

## 2 Preliminaries

### 2.1 Graphs

For prerequisites, the reader is expected to be familiar with the basic definitions of graph theory. We only point out that in this work a *chain* is meant to be a tree  $T(V, E)$  where  $V = (x_1, x_2, \dots, x_n)$  and  $E = ((x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n))$ . In other words a chain is a tree without branches. Throughout this work only finite, undirected, simple and connected graphs are considered.

### 2.2 Planarity

The most common characterization for planar graphs is the Kuratowski Theorem. It characterizes the planar graphs in terms of forbidden graphs.

**Theorem 2.1.** (*Kuratowski 1930*) *A graph is planar if and only if it contains neither a subdivision of  $K_5$  nor a subdivision of  $K_{3,3}$ .*

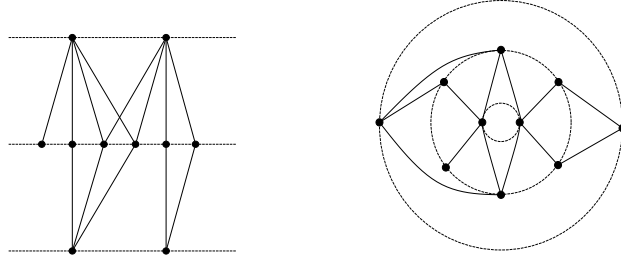
A planar graph  $G$  is said to be *maximal* if no new edge can be added without violating planarity. Thus every face of  $G$  is bounded by a triangle,  $K_3$ , and that is why it is also called a *triangulation*.

### 2.3 Level Graphs

A *k-level graph*  $G = (V, E, \lambda)$  is a graph with a mapping  $\lambda : V \rightarrow \{1, \dots, k\}$ ,  $k \geq 1$ , that partitions the vertex set  $V$  as  $V = V_1 \cup \dots \cup V_k$ ,  $V_j = \lambda^{-1}(j)$ ,  $V_i \cap V_j = \emptyset$  for  $i \neq j$ , such that  $\lambda(v) = \lambda(u) + 1$  for each edge  $(u, v) \in E$ . Thus in a drawing of a *k-level graph* in the plane all vertices are placed on *k* horizontal lines, representing *level*  $l_1, l_2, \dots, l_k$ , meaning  $v \in V_i$  is placed on level  $l_i = \{(x, k - i) | x \in \mathbb{R}\}$ . Edges are drawn as straight line segments only between consecutive levels. Hence a level graph  $G$  is called *level planar* if there exists a level drawing, an embedding in the plane, of  $G$  such that no edges cross except at their common endpoints placed on levels.

### 2.4 Radial Graphs

Radial graphs are a generalization of level graphs. A *k-radial graph* partitions the vertex set on *k radii*, which can be pictured as concentric circles, instead of levels. So the vertices are no longer spread on horizontal lines matching levels but on circles  $l_i = \{(i \cos \alpha, i \sin \alpha) | \alpha \in [0, 2\pi)\}$ ,  $1 \leq i \leq k$ . By melting the endpoints of each level of a level drawing we achieve the described concentric circles,



**Figure 1:** A level non planar embedding and a radial planar embedding of the same graph.

called radii, and therefore a radial drawing. This procedure creates some imaginary *cut ray* from the radii's center towards infinity whose intersection with the radii represent the levels' connection points. It also offers a new passage for edges to take which has not been accessible before and eventually avoids crossings. This time edges can be drawn as strictly monotone curves from inner to outer levels, but once again only between consecutive levels. Note that the term *level* is used to describe the vertex set partition in the level as well as in the radial case. Corresponding to level planarity a graph  $G$  is called *radial planar* if there exists a radial embedding, called radial drawing, of  $G$  such that no edges cross except at common endpoints.

Edges are only allowed between consecutive levels hence crossings can only appear between two neighbored levels. Looking at a drawing of a level graph  $G$ , with vertices placed on horizontal lines, we are able to detect a crossing of two disjunct edges by the position of their vertices with respect to this particular drawing. Let  $e_1 = (u_1, w_1)$  and  $e_2 = (u_2, w_2)$  be two edges with  $u_1, u_2$  on level  $l_i$  and  $w_1, w_2$  on level  $l_{i+1}$ . Edges  $e_1$  and  $e_2$  do not cross if and only if  $u_1$  is to the left of  $u_2$  on  $l_i$  and  $w_1$  to the left of  $w_2$  on  $l_{i+1}$  at the same time, or vice versa. Randerath et al. formulated this approach as a  $2CNF$ -formula and answered the question of level planarity by solving the satisfiability problem, see [8].

A radial drawing of a graph  $G$  does not provide this intuitive ordering of left and right since it is hard to say whether a vertex is placed to the right or to the left of another vertex on the same radian. Nevertheless, we are able to define an orientation with the help of the above mentioned *cut ray* as follows. By starting at the *cut ray* one can either follow the radii clock or counter clock wise for all radii and therefore get a sequence of vertices on each radian,  $(v_1, v_2, \dots, v_{n_i})$  on level  $l_i$  with  $|V_i| = n_i$ . So an orientation of a radial embedding can either be clockwise, if starting at the *cut ray* and list the upcoming vertices clockwise, or counter clockwise by taking the opposite direction. Hence by determining an orientation we are able to describe the placement of vertices in terms of their

position within the cycle notation.

Let there be an orientation defined on  $G$ . Consider two consecutive radii  $l_i$  and  $l_{i+1}$ . Given three vertices  $u, v, w \in V_i$ , we define an *interval*  $[u, w]$  on radian  $l_i$ , indicating the arc of  $l_i$  between  $u$  and  $w$ , starting in  $u$ , following the given orientation and coming to an end in  $w$  where  $u = (i \cos(\alpha), i \sin(\alpha))$ ,  $w = (i \cos(\beta), i \sin(\beta))$ . Depending on the chosen orientation, vertex  $v$  is either said to be *inside*,  $v \in [u, w]$ , or *outside* the interval,  $v \notin [u, w]$ . If  $v$  is outside the interval  $[u, w]$  it must be situated in the complement which is denoted by  $]w, u[$  with respect to the chosen orientation on the radian. The same procedure can be applied to edges. Let  $e_1 = (u_1, w_1)$  and  $e_2 = (u_2, w_2)$  be two non crossing edges with  $u_1, u_2 \in V_i$  and  $w_1, w_2 \in V_{i+1}$ . We define a *corridor*  $[e_1, e_2]$  to be the segment bordered by edges  $e_1$  and  $e_2$  and the belonging arcs on radian  $l_i, l_{i+1}$  respectively. Those arcs are the intervals  $[u_1, u_2]$  on  $l_i$  and  $[w_1, w_2]$  on  $l_{i+1}$  subject to the orientation. An additional edge  $e_3 = (v_1, v_2)$  with  $v_1 \in V_i, v_2 \in V_{i+1}$  is said to be *inside* the corridor  $[e_1, e_2]$  if  $v_1 \in [u_1, u_2]$  and  $v_2 \in [w_1, w_2]$ , respectively *outside* if  $v_1 \notin [u_1, u_2]$  and  $v_2 \notin [w_1, w_2]$ . Hence no crossing with the border edges  $e_1, e_2$  can occur if  $e_3 \in [e_1, e_2]$  and it is drawn as a straight line segment.

Keep in mind that in a radial drawing edges are only restricted to be monotone curves from inner to outer level. So edge  $e_3$  might as well be drawn as a monotone curve which crosses  $e_1$  and  $e_2$  in total an even amount of times if  $e_3 \in [e_1, e_2]$ . But then it might as well be drawn as a curve inside  $[e_1, e_2]$  that is why edge  $e_3 = (v_1, v_2)$  is said to be inside the corridor  $[e_1, e_2]$  if  $v_1 \in [u_1, u_2]$  and  $v_2 \in [w_1, w_2]$ . Also edge  $e_3$  might cross  $e_1$  and  $e_2$  in total an even amount of times even though it is outside the corridor. So the interesting case left is a crossing of edge  $e_3$  with edge  $e_1$  or  $e_2$  which cannot be avoided. Edge  $e_3$  causes a crossing with either  $e_1$  or  $e_2$  whenever  $v_1 \notin [u_1, u_2]$  and  $v_2 \in [w_1, w_2]$  or vice versa. So the problem arises whenever starting and endpoint of a new edge do lie in different corridors. Be aware that once again there might occur an odd amount of crossings with the border edges if  $e_3$  is drawn as monotone curve.

Note that from now on we only say level or radial graph while we are actually talking about the embedding or drawing of such a graph. As seen before a level graph can be transformed into a radial graph by melting the levels' endpoints. Especially level planar graphs are radial level planar. Hence the class of level planar graphs is a subclass of the radial planar graphs. A  $k$ -level radial graph is  $k$ -partite and especially bipartite which leads to the following observation.

**Observation 2.2.** *A radial graph has only cycles of even length.*

**Observation 2.3.** *If graph  $G = (V, E)$  is radial planar so are all its induced subgraphs  $G_{i,i+1}$  located between level  $l_i$  and  $l_{i+1}$ .*

Unless otherwise stated we assume that graph  $G = (V, E)$  satisfies the inequality  $|V| \geq 3$ . This condition has been made with respect to the number of edges in a simple graph. One edge can occur at most in a graph with less vertices and consequently no edge crossing.

## 2.5 Euler's Polyhedral Formula

Using the Eulerian Formula we can make several statements about planar graphs. Euler's Polyhedral Formula reads as follows:

$$n + f - m = 2 \quad (1)$$

where  $n$  is the number of vertices,  $f$  the number of faces and  $m$  the number of edges. It is known that Equation (1) can be used to provide a maximal number of edges in a planar graph  $G$ , [see, for instance [3]]. Applying it to Equation (1) leads to:

$$m = 3n - 6 \quad (2)$$

Hence a planar graph can have at most  $3n - 6$  edges. The class of radial planar graphs is a subclass of the class of bipartite and planar graphs. Thus only cycles of even length occur and the smallest face possible is a  $C_4$ . Applying it to Equation (1) leads to:

$$m = 2n - 4 \quad (3)$$

Hence a bipartite and planar graph, and therefore a radial planar graph, can have at most  $2n - 4$  edges. That is how we reach the criteria for radial planar graphs:

$$m \leq 2n - 4 \quad (4)$$

So we know that graph  $G$  cannot be radial planar if the inequality is violated. Note that non radial planar graphs can fulfill the inequality as well, e.g.  $K_{2,3}$ . That is why we are looking for a tighter bound for the number of edges in Section 4.

## 3 Forbidden Patterns

Based on three level non planar patterns for hierarchies by Di Battista and Nardelli, [1], Healy et al. introduced the notion of *minimal level non planar* patterns (MLNP) for level graphs, [4]. A *hierarchy* is a level graph  $G = (V, E)$

where for every  $v \in V_i, i > 1$ , there exists at least one edge  $(w, v)$  such that  $w \in V_{i-1}$ . A level graph is said to be minimal level non planar if deleting an arbitrary edge leads to level planarity. So these MLNP patterns do match the subdivisions  $K_5$  and  $K_{3,3}$  of general planar graphs. Healy et al. gave a complete characterization of level planar graphs in terms of minimal forbidden subgraphs which are identified by trees, level non planar cycles and level planar cycles with augmented paths. Kuratowski provided a characterization for planar graphs in terms of forbidden subgraphs. By the end of this section we will give a complete characterization for radial planar graphs in terms of forbidden patterns, the so called *minimal radial non planar* (MRNP) patterns.

### 3.1 Level-planarity

We use the terminology Healy et al. have used to describe the MLNP patterns [4]. A *pattern*  $P = (V', E')$ ,  $V' \subset V, E' \subset E$ , is a level or radial embedded subgraph of  $G$  which can be described by its upper- and lower-most levels, the so called *extreme levels* of  $P$ . If a vertex  $v$  is located on an extreme level then this one is called the *incident extreme level*. The other extreme level is called the *opposite extreme level* of  $v$ .

**Theorem 3.1.** (Healy, Kuusik and Leipert [4]). *The set of MLNP patterns characterized by trees, LNP cycles and path-augmented level planar cycles is complete for hierarchies.*

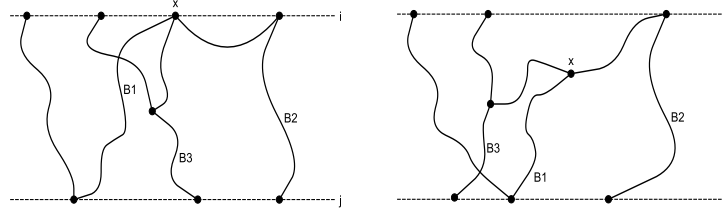
#### 3.1.1 Trees

Healy et al. characterized MLNP trees as follows, see [4]. Let  $x$  denote a root vertex with degree three which is located on one of the levels  $l_i, \dots, l_j$ . There have to be three subtrees, which emerge from the root vertex, that have the following common properties:

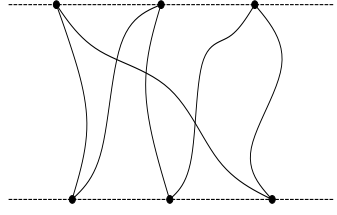
- Each subtree has at least one vertex on both extreme levels.
- A subtree is either a chain or it has two branches which are chains.
- All the leaf vertices of the subtrees are located on the extreme levels, and if there is a leaf vertex  $v$  of a subtree  $S$  on an extreme level  $l \in \{i, j\}$  then  $v$  is the only vertex of  $S$  on  $l$ .
- Those subtrees which are chains have one or more non-leaf vertices on the extreme level opposite to the level of their leaves.

According to the location of the root vertex two instances occur:

- (T1) The root vertex  $x$  is on an extreme level  $l \in \{i, j\}$ . At least one of the subtrees is a chain starting from  $x$ , going to the opposite extreme level of  $x$  and finishing on  $x$ 's level.
- (T2) The root vertex  $x$  is on one of the intermediate levels  $l, i < l < j$ . At least one of the subtrees is a chain that starts from  $x$ , goes to the extreme level  $l_i$  and finishes on level  $l_j$ . Furthermore, at least one of the subtrees is a chain that starts from  $x$ , goes to level  $l_j$  and finishes on level  $l_i$ .



**Figure 2:** Minimal level non planar tree pattern T1 and T2.



**Figure 3:** Minimal level non planar pattern, LNP cycle.

### 3.1.2 LNP cycles

A level non planar (LNP) cycle is a cycle bounded by the extreme levels  $l_i$  and  $l_j$ . In contrast to level planar cycles which consists of two distinct paths between the extreme levels MLNP cycles must contain at least four distinct paths between the extreme levels having only endpoints on the extreme levels. Such a path is called a *pillar*.

### 3.1.3 Path-augmented cycles

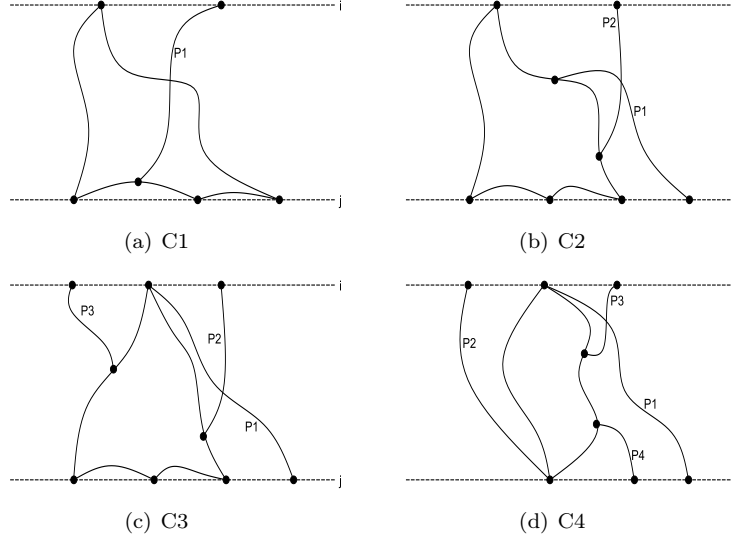
A level non planar pattern including a cycle can also be achieved as a consequence of augmenting a level planar cycle by one or more paths. In order to specify the minimal path-augmented level non planar cycles we need some further definitions which can be found in [4]. As mentioned before a cycle has at



least two pillars. Vertices situated on pillars are called *outer* vertices where all others are called *inner* vertices. A pillar's endpoint is named *corner* vertex, which is also called *single corner* if it is the only vertex on the extreme level. A *bridge* is said to be the shortest path between corner vertices on the same level. This is as much to say as a bridge has two corner vertices as its endpoints and all remaining vertices are inner ones. A pillar is *monotonic* if the level numbers of consecutive vertices of the pillar are either monotonically increasing or decreasing. The *starting* vertex of a chain is denoted by the vertex of degree one, considering only the chain's vertices, which is connected to a cycle, the other vertex of degree one is the *ending* vertex.

According to [4] a minimal level non planar path-augmented cycle has to be one of the four cases. The augmented paths always start at a vertex of the cycle and end on an extreme level. Let  $l_i$  and  $l_j$  be the bounding extreme levels.

- (C1) A single path  $p_1$  starting from an inner vertex  $v_{p1}$  and ending on the opposite extreme level of the inner vertex;  $p_1$  and the cycle only have the vertex  $v_{p1}$  in common. The path has at least one vertex on an extreme level, the end vertex, and at most two, the start and end vertices.
- (C2) Two paths  $p_1$  and  $p_2$ , starting, respectively, from vertices  $v_{p1}$  and  $v_{p2}$ ,  $v_{p1} \neq v_{p2}$ , of the same pillar  $L = (v_i, \dots, v_{p1}, \dots, v_{p2}, \dots, v_j)$  terminating on extreme levels  $l_j$  and  $l_i$ , respectively. Vertices  $v_{p1}$  or  $v_{p2}$  may be identical to corner vertices of  $L$  ( $v_{p1} = v_i$  or  $v_{p2} = v_j$ ) only if the corner vertices are not single corner vertices on their extreme levels. Path  $p_1$  and  $p_2$  have no vertices other than their start (if corner) and end vertices on the extreme levels. There are two subcases according to the levels of  $v_{p1}$  and  $v_{p2}$ :  $\lambda(v_{p1}) < \lambda(v_{p2})$  or  $\lambda(v_{p1}) \geq \lambda(v_{p2})$ . The latter means that  $L$  must be a non-monotonic pillar.
- (C3) Three paths  $p_1, p_2$  and  $p_3$ . Path  $p_1$  starts from a single corner vertex and ends on the opposite extreme level; paths  $p_2$  and  $p_3$  start from opposite pillars and end on the extreme level where the single corner vertex is at. Neither  $p_2$  nor  $p_3$  can start from a single corner vertex.
- (C4) Four paths  $p_1, p_2, p_3$  and  $p_4$ . The cycle comprises a single corner vertex on each of the extreme levels. Paths  $p_1$  and  $p_2$  start from different corner vertices and end on the opposite extreme level to their start with the paths embedded on either side of the cycle such that they do not intersect; paths  $p_3$  and  $p_4$  start from distinct non-corner vertices of the same pillar and finish on different extreme levels. The level numbers of starting vertices are such that they do not cause crossing of the last two paths.



**Figure 4:** Minimal level non planar pattern, path-augmented level planar cycles C1, C2, C3 and C4.

### 3.2 Radial-planarity

In Section 3.1 we have seen a complete set of minimal level non planar patterns. We want to apply this characterization to radial graphs now in order to find one for minimal radial non planar patterns.

Let  $G$  be a radial graph with extreme levels  $l_i$  and  $l_j$  of the considered pattern. Assume we defined an orientation on  $G$ . We therefore know that two edges do not cross within a chosen corridor, if their starting and endpoints appear in the same order according to this orientation.

So far a corridor has only been defined for two consecutive levels. We now generalize it to spread over more than two levels bounded by the extreme levels  $l_i$  and  $l_j$ . Let  $p_1, p_2$  be two distinct pillars which have their starting and endpoints on the extreme levels  $l_i$  and  $l_j$ . By definition  $p_1, p_2$  do have a vertex on every level  $l_t$ ,  $i < t < j$ . That is to say a *corridor*  $[p_1, p_2]$  is bounded by two distinct pillars  $p_1, p_2$  and the arcs on radii  $l_i$  and  $l_j$  defined by the pillars' starting and endpoints according to the chosen orientation. So a corridor is no longer restricted to contain only edges but subgraphs. Likewise edges a path  $p_3$  is said to be inside the corridor,  $p_3 \in [p_1, p_2]$  if every edge of  $p_3$  is inside the corridor built by the belonging edges of  $p_1$  and  $p_2$ . Note that once again a path might have an even amount of crossings with the border paths and is still said to be inside the corridor where as an odd amount of crossings results in a path having

either starting or endpoint outside the considered corridor, the same way as it was outlined for edges.

### 3.2.1 Trees

Taking the minimal level non planar tree patterns and regarding them in radial graphs with vertices distributed on radii and no longer on levels, we prove that they are minimal radial non planar as well. So the set of MLNP trees and the set of MRNP trees can be set equal without adding or deleting any properties. Note that we are still talking about levels even though the vertices are now spread on concentric circles and no longer on horizontal lines.

**Theorem 3.2.** *MLNP trees are minimal radial non planar patterns (MRNP).*

*Proof.* We prove that the MLNP tree patterns T1 and T2 are minimal radial non planar.

(T1): Looking at an arbitrary MLNP tree pattern  $P$  with extreme levels  $l_i$  and  $l_j$  that fulfills condition T1, let  $B_1$  and  $B_2$  be two of the three required subtrees. They can be embedded without any crossing. W.l.o.g. let root vertex  $x$  be situated on level  $l_i$ . Our aim is to construct an interval  $[b_1, b_2]$  on level  $l_i$  where  $b_1$  is a vertex of  $B_1$ ,  $b_2$  a vertex of  $B_2$  and  $x \in [b_1, b_2]$ . Vertices  $b_1, b_2$  do exist by assumption since every subtree has at least one vertex on each extreme level. In order to label vertices  $b_1$  and  $b_2$  we direct edges. Let root vertex  $x$  be the starting point and direct all adjacent edges away from  $x$ . Whenever we reach a new vertex, we repeat the action and direct all adjacent edges, that have not been directed so far, away from the vertex. After doing so, choose the longest directed path, that starts in  $x$  and ends on radian  $l_j$ . Repeating it we obtain two vertices  $b_1 \in B_1$  and  $b_2 \in B_2$  which are the desired endpoints of our interval  $[b_1, b_2]$ . Without loss of generality let the orientation be chosen in such a way that  $x \in [b_1, b_2]$  holds. Otherwise choose the opposite orientation.

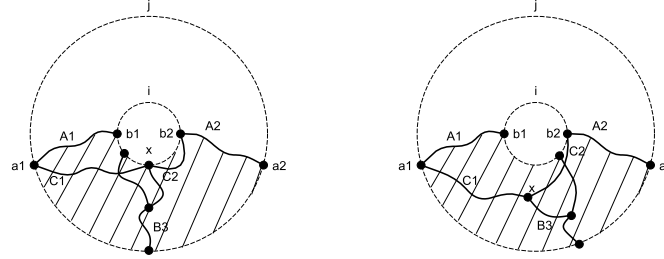
For the same reason there have to be two vertices  $a_1, a_2 \in V_j$  on radian  $l_j$  with  $a_1$  part of  $B_1$  and  $a_2$  part of  $B_2$ , such that  $a_k$  ( $k = 1, 2$ ) is the endpoint of a path  $A_k$  that starts in  $b_k$  on radian  $l_i$  and goes straight to radian  $l_j$  without passing vertex  $x$ . Thus  $A_k \subset B_k$  is a path between radian  $l_i$  and  $l_j$ , hence a pillar. We can now build the corridor  $[A_1, A_2]$  bordered by the two pillars  $A_1, A_2$  and  $x \in [A_1, A_2]$ . Since path  $A_k$  is part of the subtree  $B_k$  but  $x \notin A_k$ , there exists another path  $C_k$  such that  $C_k = B_k - A_k$ . Path  $C_k$  therefore starts in root vertex  $x$  and ends either in an endpoint of  $A_k$  or an inner vertex of  $A_k$  depending on  $B_k$  being a chain or a subtree with two branches, see Figure 5.

Now looking at pattern  $P$  we still have to put the third requested subtree  $B_3$  with root vertex  $x$  in place. Since  $x \in [A_1, A_2]$  all further subtrees starting in

$x$  have to be embedded in the corridor as well in order to not cause a crossing, also  $B_3$ . By assumption  $B_3$  has to have at least one vertex, other than the root vertex  $x$ , on each extreme radian. But placing subtree  $B_3$  in  $[A_1, A_2]$  leaves no possibility for a path from level  $l_i$  to  $l_j$  without crossing either  $C_1$  or  $C_2$ . Hence T1-patterns are radial non planar as well.

(T2): Given an arbitrary MLNP tree pattern  $P$  bounded by extreme levels  $l_i$  and  $l_j$  that fulfills condition T2, we know that there has to be at least one chain starting in root vertex  $x$ , going to radian  $l_j$  and ending on  $l_i$  and vice versa. Using this information the same reasoning as in the case T1 applies.

The set of radial planar graphs is an upper set of the set of level planar graphs. Patterns T1 and T2 were minimal level non planar. Thus deleting an arbitrary edge especially leads to a radial planar pattern. So Theorem 3.2 has been proved.  $\square$



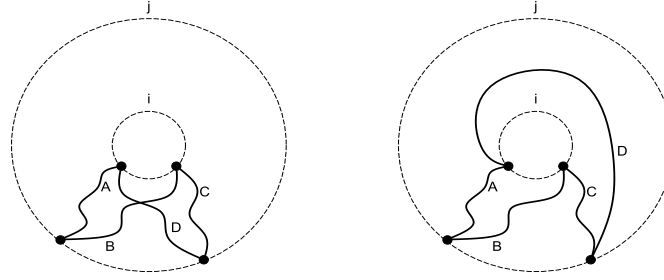
**Figure 5:** Minimal radial non planar tree pattern T1 and T2.

### 3.2.2 Cycles

**Radial planar cycles** The pattern of LNP cycles gives us a first impression of patterns which are not level planar but radial planar. Taking the radial planar model by merging levels to radii we gain an extra possibility to place an edge *around* the inner radian which leads to radial planarity.

**Lemma 3.3.** *LNP cycles are radial planar.*

Lemma 3.3 may be proved in much the same way as it has been in the level non planar case in [4] except the fact that the last pillar crosses the imaginary cut ray which evolves from melting a levels' endpoints and not the remaining pillars, see Figure 6. That is why this realization has not been possible in the level planar case.



**Figure 6:** A radial non planar embedding of the level non planar cycle  $C_4$  to the left and a radial planar embedding to the right.

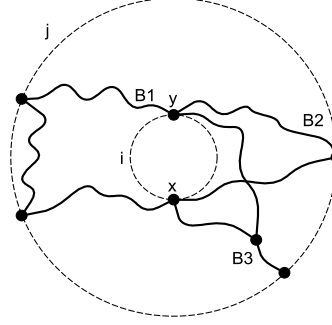
**Radial Non Planar cycles (RNP)** We have seen above that LNP cycles are no candidates for radial non planar patterns. Instead they are radial planar. But they can be achieved by melting two minimal radial non planar tree patterns. That property makes them convenient as a source of further minimal radial non planar patterns including a cycle which is gained by melting the trees' leaf vertices and an additional path.

Given a LNP cycle, there are three possibilities to start an additional path in order to build a MRNP pattern. The path's starting and end vertices can either be corner vertices and lie on a shared extreme radian, they can both be non corner vertices on intermediate radii such that the path starts and ends at inner vertices of different pillars or the path can have starting and endpoint on two different extreme radii.

We will have a closer look at the case of an additional path starting and ending on extreme radian  $l_i$  and having at least one vertex on the extreme radian  $l_j$ , for an example see Figure 7. We will prove that cycles augmented by such a *loop*, which will be characterized later on, are radial non planar. We take the characterization T1 of Healy et al for MLNP trees and slightly adjust it to achieve a characterization for MRNP cycles. Note that the additional path has to end in the melted leaf vertex on radian  $l_i$  otherwise the pattern would not be minimal.

- **(C1T1):** Additionally to T1, all three subtrees end in one shared leaf vertex  $y$  on radian  $l_i$ . Hence, only a subtree consisting of two branches can have a leaf vertex other than  $y$ .

The assumption of a shared leaf vertex  $y$  on radian  $l_i$ , in which point all three subtrees merge, leads towards the picture of a cycle with a loop. That is why we call the third subtree respectively the path a *loop*. Note that we are still talking about subtrees even though they build a cycle.



**Figure 7:** An example of a MRNP pattern C1T1, a cycle with an additional path at the inner radian.

**Lemma 3.4.** *Loop-augmented cycles as described by C1T1 are MRNP patterns.*

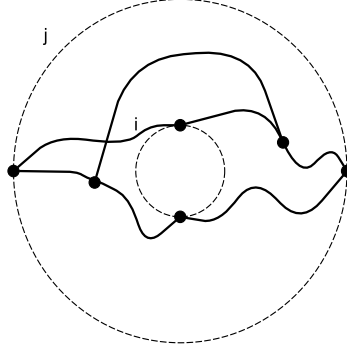
*Proof.* The proof can be adapted from Theorem’s 3.2 proof. The only difference is that in this case pillars  $A_1$  and  $A_2$  have their starting point  $y$  in common. Besides that radial non planarity can be proven using the same arguments as in 3.2.

The fact of minimality is still missing. Since we did not force any further restrictions on  $B_3$ , apart from C1T1, any of the three subtrees could have been  $B_3$ . Thus it suffice to show that deleting any edge from  $B_3$  creates a radial planar pattern. We only have to distinguish between  $B_3$  being a chain or having two branches which are chains.

**Chain:** Deleting an arbitrary edge implies two paths  $P_1$  and  $P_2$  (one possibly empty). Let  $P_1$  be the one starting in  $x$  and  $P_2$  the one starting in  $y$ .  $P_1$  can be embedded inside  $[A_1, A_2]$  since it starts in  $x$  but does not have a leaf vertex in  $y$  any longer and therefore does not have to cross  $C_k, k = 1, 2$ . And  $P_2$  can be embedded outside  $[A_1, A_2]$  since it starts in  $y$  but has no further vertices on radian  $l_i$ .

**Branches:** Deleting an edge on the way from  $x$  or  $y$  to the branching point reduces to the above chain. On the other hand, deleting an edge between the branching point and the leaf vertex on radian  $l_j$  implies a smaller subtree with two branches that can be embedded inside the chain subtree  $B_1$  or  $B_2$  which must exist by assumption.  $\square$

Further augmentations are described by the upcoming patterns C2T1 and C3T1 which either consists of a radial planar cycle with a loop starting at one of the

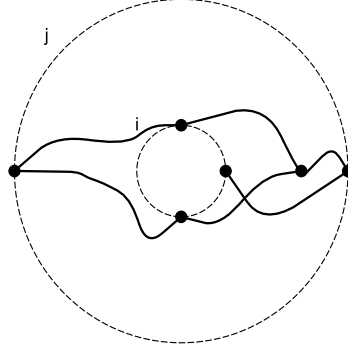


**Figure 8:** An example of the MRNP pattern C2T1.

intermediate radii or a path that starts on an extreme radian and ends on the opposite one, see Figures 8 and 9.

- **(C2T1):** Adjust T1 by melting the leaf vertices of two subtrees that are direct chains. The third subtree is transformed into a loop that starts on an intermediate radian  $l_s$ ,  $i < s < j$ , of a pillar. The loop is supposed to end on an intermediate radian  $l_e$ ,  $i < e < j$ ,  $l_e \neq l_s$ , of the one pillar that does not share a corner vertex with the pillar the starting vertex is part of.
- **(C3T1):** Adjust T1 by melting the leaf vertices of two subtrees  $B_1, B_2$  into a vertex  $y$  on radian  $l_i$ . Besides, let subtree  $B_1$  be a direct chain and subtree  $B_2$  have two branches which are chains. An additional path starts from  $B_2$ 's leaf vertex  $p$  on radian  $l_j$  and ends on radian  $l_i$ .

Note that the structure described by C3T1 is the only possibility to gain a minimal pattern with a radial planar cycle having two corner vertices on every extreme radii and augmented by a path having starting and end vertex on different extreme radii. Notice that the additional path can have no vertices other than its starting and end vertices on an extreme radian. Since we aimed for a MRNP pattern a crossing is necessary. That is the reason why the augmented path can only end in an additional vertex on radian  $l_i$  which is not part of the radial planar cycle. The only way the cycle can be constructed is by melting a subtree which represents a chain and a subtree with two branches which are chains. Any other combination would either not be minimal or not cause a crossing at all.



**Figure 9:** An example of the MRNP pattern C3T1.

### 3.2.3 Path-augmented Cycles

As we will see in this section, besides the LNP cycles, the minimal level non planar path-augmented cycles are a further class of patterns being level non planar but radial planar. Again, we show that these patterns can be modified in a way that they do become minimal radial non planar as well and we therefore will have found another class of MRNP patterns.

#### Radial planar path-augmented cycles

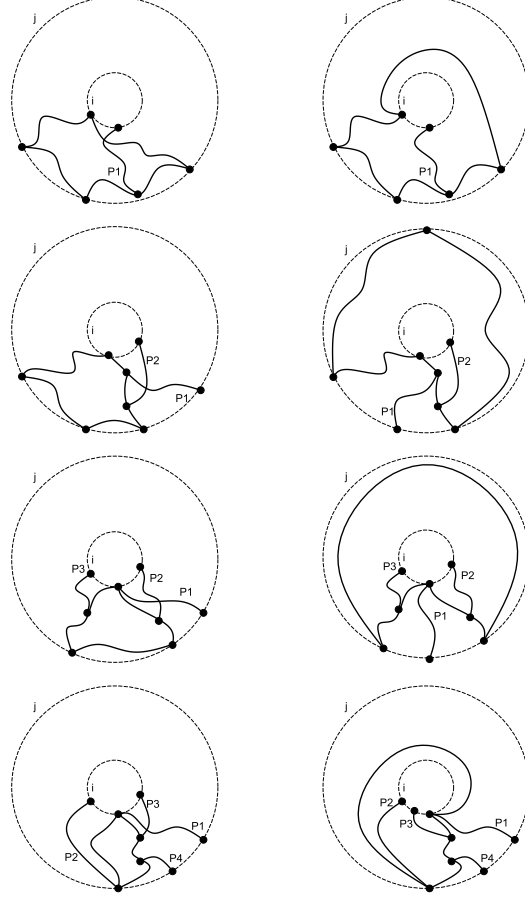
**Lemma 3.5.** *Path-augmented cycles are radial planar.*

*Proof.* We need to show that all four patterns C1, ..., C4 are radial planar. We give the proof only for the case C1. The same conclusion can be drawn for C2, C3 and C4 and will appear in a forthcoming publication. Figure 10 shows a radial planar embedding of all four patterns. Let  $l_i, l_j$  be the extreme radii of the considered pattern in a level planar embedding.

(C1): Without loss of generality, let  $l_i$  be the radian path  $p_1$  ends on. Furthermore, let  $A_k$ ,  $k = 1, 2$  be the pillars of the level planar cycle with corner vertices  $a_k \in V_j$ . Taking the pillars into account we are able to build the corridor  $[A_1, A_2]$ . Since  $p_1$  starts from an inner vertex, there has to be a bridge  $C$  going from  $a_1$  to  $a_2$  without having any vertices on radian  $l_i$ . Now choose an orientation such that  $C \in [A_1, A_2]$  holds. In the level case the crossing between  $p_1$  and the level planar cycle was caused since  $p_1$  as well as the two bridges of the level planar cycle had to be in between  $A_1$  and  $A_2$ . In the radial case we are now able to place bridge  $C$  outside  $[A_1, A_2]$  by letting it run around the inner radian  $l_i$ . Note that outside the corridor  $[A_1, A_2]$  means inside the complement, the corridor  $[A_2, A_1]$ , according to the chosen orientation. So path  $p_1$ , starting



from an inner vertex on  $C$ , can be placed outside  $[A_1, A_2]$  as well and reach radian  $l_i$  without causing a crossing. Hence, pattern C1 is radial planar.  $\square$



**Figure 10:** MLNP patterns C1, C2, C3, C4 to the left and their radial planar embeddings to the right.

**Radial non planar path-augmented cycles** Having proved that patterns C1, C2, C3 and C4 are radial planar we are led to the question if corresponding augmentation of these patterns exist that create minimal radial non planar patterns. Following the method we have used for LNP cycles, we augment our path  $p_k$  to be a subtree again with the properties recommended for radial non planar tree patterns. Any other augmentation cannot be minimal since it should have

occurred somehow as a pattern in the level planar case. Hence  $p_k$  can either be augmented to be a chain with a bridge, a direct chain with exactly one vertex on each extreme radian or to have two branches which are chains.

**Lemma 3.6.** *In order to receive a MRNP pattern based on the radial planar patterns  $C1$ ,  $C2$ ,  $C3$  or  $C4$ , path  $p_k$  cannot be augmented to have two branches which are chains or to be a chain with a bridge.*

*Proof. Branches:* Augmenting  $p_k, k = 1 \dots, 4$ , to be a subtree with two branches which are chains and root vertex  $p_k$  in pattern  $(Cl)$ ,  $l = 1, \dots, 4$  does not lead to a minimal radial non planar pattern because one can always find an induced tree pattern with root vertex  $x = v_{pk}$ . Induced tree pattern is briefly meant for induced minimal radial non planar tree pattern. We might have to adjust the extreme radii since we are looking at an induced pattern, for example take the radian the branching point or one of the starting vertices  $v_{pk}$  is situated on as the belonging extreme radian and reduce the range of the pattern. Then deleting any arbitrary edge, which does not belong to the induced tree pattern  $(\tilde{Cl})$  with the smaller level span, does not provide a radial planar pattern. Thus, augmenting  $p_k$  to be a subtree with two branches which are chains in pattern  $(Cl)$ ,  $l = 1, \dots, 4$  does not provoke a MRNP pattern.

**Bridge:** Assume  $p_k$  has been augmented to be a chain with a bridge  $B$  on radian  $l_i$  in pattern  $(Cl)$ ,  $l = 1, \dots, 4$ . The resulting pattern is not minimal because deleting an arbitrary edge of  $B$  induces a  $(Cl)$  pattern with an additional path from radian  $l_i$  to  $l_j$ . This path cannot be embedded along with the  $(Cl)$  pattern without causing a crossing since  $(Cl)$  can only be embedded radial planar by placing a cycle around the inner radian as seen before. This prevents us from embedding the second component, the path from radian  $l_i$  to  $l_j$ , otherwise a crossing would be inevitable. The same conclusion can be drawn for  $B$  located on radian  $l_j$ .  $\square$

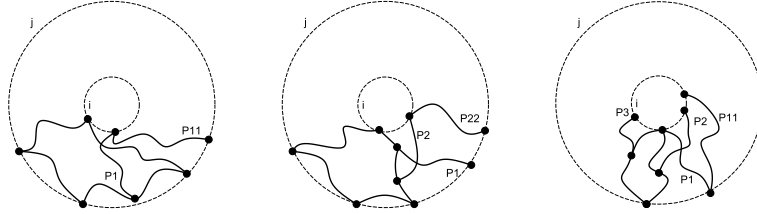
Hence, the only possibility left is augmenting  $p_k$  to be a direct chain with exactly one vertex on each extreme radian. The following argumentation shows that it actually leads towards minimal radial non planar patterns. So assume  $p_k$  has been augmented in the mentioned way.

**Augmented (C1):** Let  $l_i$  be the radian the original not augmented path  $p_1$  ended on. Since  $p_1$  starts from an inner vertex  $v_{p1}$ , there has to be a bridge  $B_j$  on radian  $l_j$ . Let  $p_1$  no longer be just a path to radian  $l_i$  but a subtree that has vertices on both extreme radii. Now assume we also have a bridge  $B_i$  on radian  $l_i$ . Deleting an arbitrary edge of  $B_i$  would only now cause a radial non

planar tree pattern with root vertex  $v_{p_1}$ . Therefore a bridge  $B_i$  cannot exist in a minimal radial non planar pattern.

According to Lemma 3.6 we can only augment  $p_1$  to a chain with exactly one vertex on each extreme radian. Let  $p_{11}$  be the augmentation of  $p_1$  which goes from radian  $l_i$  to radian  $l_j$ . Subject to the proof of Lemma 3.5 pattern C1 can only be embedded radial planar by placing a cycle around the inner radian. This can either be done using a pillar or eventually a bridge. Nevertheless any further augmentation of path  $p_1$  to be a chain with vertices on both extreme radii forces a crossing. Thus, the augmented C1 pattern is radial non planar.

We still need to show minimality. Deleting an edge of one of the radial planar cycle's pillars results in two paths. The one starting on radian  $l_i$  can be placed next to the other pillar without crossing  $p_1$  any more. Respectively, one can embed  $p_1$  in the occurring gap of the pillar. By deleting an edge from bridge  $B_j$  we are able to embed the part of  $B_j$  with  $p_1$  on it next to the former level planar cycle such that  $p_1$  and its augmentation can be placed next to the level planar cycle without crossing it any longer. Deleting an edge from  $p_1$  implies one half of  $p_1$ , which can be embedded inside the cycle, and  $p_{11}$  with the other half of  $p_1$  embedded outside. In the end deleting an edge from  $p_{11}$  causes a radial planar pattern since C1 is being radial planar. The only difference is an additional path which does not go from radian  $l_i$  to  $l_j$ , therefore no crossing occurs. Hence, the only way of augmenting  $p_1$  in order to achieve a minimal radial non planar pattern is to extend it to a chain with exactly one vertex on each extreme radian.



**Figure 11:** Examples of minimal radial non planar (MRNP) augmented cycle patterns, Augmented C1, C2 and C3.

**Augmented (C2)-(C3):** Similar arguments apply to the case of Augmented C2 and Augmented C3 and it can be shown that those patterns are minimal radial non planar as well.

**Augmented (C4):** Augmenting either one of the four paths  $p_k$ ,  $k = 1, \dots, 4$ , leads to an induced tree pattern with root vertex  $x = v_{p_k}$ . So the augmented C4 pattern cannot be minimal.

**Lemma 3.7.** *Patterns caused by augmenting a path of **C1**, **C2** or **C3** to be a direct chain with exactly one vertex on each extreme radian and following the rules of **Augmented C1**- **Augmented C3** are minimal radial non planar.*

Based on the MLNP patterns introduced in [4] we have found three classes of MRNP patterns. First, there are the tree patterns which are minimal radial non planar. In particular, those are the only minimal level non planar patterns that are radial non planar as well. The reason for this can be found in the tree structure which cannot take advantage of the possibility to place an edge around an inner radian and thus avoid edge crossings. This is only relevant for cycles not for trees. Second, there are the RNP cycles with at least four pillars and an augmented loop. They are a generalization of the LNP cycles. Last but not least the radial non planar path-augmented cycles which are an extension of level non planar path-augmented cycles. They consist of a cycle with two pillars and up to four paths whereat one of them is augmented to have exactly one vertex on each extreme radian.

According to [4] the set of MLNP patterns is complete for hierarchies. Compared to level graphs the radial graphs offer the opportunity to embed a cycle around a radian, it can be wrapped around. We have used this property and considered all possible augmentations. So if there exists a minimal radial non planar pattern it must match one of the patterns mentioned above. Any other augmentation should have occurred in the level case as well. We can now formulate our main result.

**Theorem 3.8.** *Let  $G = (V, E, \lambda)$  be a hierarchical radial graph then  $G$  is radial planar if and only if it contains none of the MRNP patterns described by  $T1, T2, C1T1-C3T1$  and **Augmented C1-C3**.*

## 4 Preprocessing

Before running an algorithm to test for planarity it makes sense to check the satisfiability of some constraint concerning the number of edges in a radial planar graph. From now on let  $G = (V, E)$  be a  $k$ -radial-planar graph with  $|V| \geq 3$ .

### 4.1 An upper bound for the number of edges

The sum of all edges in between each two levels sums up to the entire amount of edges. Let  $G$  be a level graph with  $k$  levels. Let  $V_i \subset V$ , with  $|V_i| = n_i$ , be the vertices on level  $l_i$  and  $m_{i,i+1}$  the number of edges between level  $l_i$  and  $l_{i+1}$

where  $i = 1, \dots, k-1$ . Thus  $\sum_{a=1}^k n_a = |V| = n$  and  $\sum_{i=1}^{k-1} m_{i,i+1} = |E| = m$ .

From now on we will be looking at the subgraphs of  $G$  with level span 1, that are the induced subgraphs  $G'$  given by vertices  $V_i$  on level  $l_i$  and  $V_{i+1}$  on  $l_{i+1}$ . Let  $G$  be radial planar. The difference between level and radial planarity are the cycles.  $G'$  can either contain a cycle or it is a tree. These two cases are to be investigated.

#### 4.1.1 Cycle

Assume there is a longest cycle  $C_c$  in  $G'$ . Since  $G$  is radial planar so is  $G'$  and  $C_c$  has to be of even length  $c$ ,  $c \geq 4$ . Thus there are at least  $\frac{c}{2}$  vertices on level  $l_i$ . The same holds for level  $l_{i+1}$ . The remaining  $n_i - \frac{c}{2}$  vertices on  $l_i$  which are not part of  $C_c$  can either have degree one or zero in  $G'$ . A greater degree would induce one of the MRNP in  $G'$  and therefore cause a crossing. These vertices of degree one will be called leaf vertices and together with the adjacent edge that connects to one of the cycle's vertices it is called a branch. A cycle vertex can have an arbitrary amount of branches as long as those branches are situated in between the two cycle chords adjacent to the cycle vertex. Those chords do have the same boundary character as the paths in Section 3.2.2. This gives reason for the following lemma.

**Lemma 4.1. Degree Condition Cycle:** *Let  $G'$  be the radial planar subgraph induced by the vertices  $V_i$ ,  $|V_i| = n_i$ , on level  $l_i$  and the vertices  $V_{i+1}$ ,  $|V_{i+1}| = n_{i+1}$ , on level  $l_{i+1}$ . If there exists a longest cycle  $C_c$  of even length  $c$ ,  $c \geq 4$ , in  $G'$  then the following holds:*

- the  $c$  cycle vertices have degree  $\geq 2$
- the  $n_i + n_{i+1} - c$  remaining non-cycle vertices have degree  $\leq 1$ .

*Proof.* Assume there is a cycle vertex  $v$  with  $\deg(v) \leq 1$ . This is a contradiction since a cycle vertex has to have degree at least 2. Also assume there is a non-cycle vertex  $v$  with  $\deg(v) \geq 2$ . This will lead to a radial level non planar pattern. If  $c = 4$  there is the pattern of a  $C_4$  with an augmenting path, which is radial level non planar. If  $c \geq 6$  one will receive an induced radial level non planar tree pattern. Hence a contradiction as well.  $\square$

Note that as soon as there exists a cycle in  $G'$  there can only be one connected component which consists of more than one vertex since a cycle and an edge which is not part of the cycle's component would always cross. In the following the word component means connected component. Let  $\mathcal{C}_{G'}$  be the set of components in  $G'$ .

**Case**  $|\mathcal{C}_{G'}| = 1$ : There is only one component  $\mathcal{C}_1$  which consists of the cycle  $C_c$  and possibly several branches. We now want to count the edges in  $\mathcal{C}_1$ . There are  $c$  edges coming from  $C_c$ . All non-cycle vertices have degree one and no other edges than the branches can occur. So we have  $(n_i + n_{i+1}) - c$  edges belonging to the branches. Hence the total number of edges in  $\mathcal{C}_1$  is exactly

$$|E_1| = c + (n_i + n_{i+1}) - c = n_i + n_{i+1}. \quad (5)$$

**Case**  $|\mathcal{C}_{G'}| \geq 2$ : We have already seen that apart from the component which contains  $C_c$  all other components consist of the complete graph with one vertex, the  $K_1$ , and no further edges. So the amount of edges in  $G'$  depends on the amount of edges in the component containing  $C_c$  while taking into account the number of  $K_1$ 's. Hence we achieve the **Edge Condition**:

$$|E'| = c + (n_i + n_{i+1}) - c - |K_1| = n_i + n_{i+1} - |K_1| \quad (6)$$

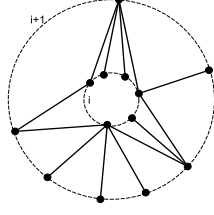
The degree condition provides a cycle with possible branches and maybe not connected single edges or vertices while the edge condition guarantees the existence of maximal one connected component with edges. Thus we can summarize:

**Theorem 4.2.** *Let  $G$  be a graph with the induced subgraph  $G'$  between level  $l_i$  with vertices  $V_i$ ,  $|V_i| = n_i$ , and level  $l_{i+1}$  with vertices  $V_{i+1}$ ,  $|V_{i+1}| = n_{i+1}$ . Let  $C_c$  be a longest cycle of even length  $c$ ,  $c \geq 4$ , in  $G'$ .  $G'$  is radial planar if and only if*

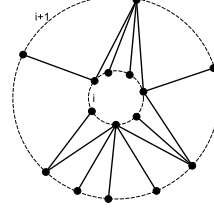
- *the  $c$  cycle vertices have degree  $\geq 2$*
- *the  $n_i + n_{i+1} - c$  remaining non-cycle vertices have degree  $\leq 1$*
- $|E'| = n_i + n_{i+1} - |K_1|$

*Proof.* ' $\Rightarrow$ ' Assume  $G'$  is radial planar. According to Lemma 4.1 and the fact that there can only be one component with more than one vertex the Degree and Edge Conditions hold.

' $\Leftarrow$ ' Assume the Degree and Edge Conditions hold. In the component of the  $C_c$  we are able to place all non-cycle vertices in between the cycle chords of the cycle vertex they are connected to. So far the component is radial planar. Furthermore we claim that there can only be components consisting of one vertex and no edge. Adding an edge in a component other than the one with the  $C_c$  would raise the number of edges on the left side of the Edge Condition by one. But at the same time the number of vertices on the right side of the Edge Condition would be raised by two while no further  $K_1$  component is subtracted. So the Edge Condition would be violated, a contradiction.  $\square$



**Figure 12:** An induced subgraph  $G'$  with a longest cycle  $C_6$ .



**Figure 13:** An induced subgraph  $G'$  with a longest path  $P_8$ .

Note that as soon as a longest cycle  $C_c$  of length  $c$  is found all other vertices not belonging to  $C_c$  are regarded as non-cycle vertices, even if they are part of another cycle.

#### 4.1.2 Tree

Now assume there is a longest induced path  $P_p$  of length  $p - 1$ ,  $p \geq 2$ , in  $G'$  and no cycle, saying  $G'$  is a forest. Once again since  $G$  is radial planar so is  $G'$ . Just as before let  $\mathcal{C}_{G'}$  be the set of connected components in  $G'$ .

**Case  $|\mathcal{C}_{G'}| = 1$  :** There is only one component  $\mathcal{C}_1$  which consists of a longest path  $P_p$  of length  $p - 1$  and possibly more edges. What can be said about the vertices' degrees in the component? We adjust the degree condition for the case of a path.

**Lemma 4.3. Degree Condition Path** *Let  $G'$  be the radial planar subgraph induced by the vertices  $V_i$ ,  $|V_i| = n_i$ , on level  $l_i$  and the vertices  $V_{i+1}$ ,  $|V_{i+1}| = n_{i+1}$ , on level  $l_{i+1}$ . If there exists a longest path  $P_p$  of length  $p - 1$ ,  $p \geq 2$ , in  $G'$  and no cycle then the following holds:*

- the  $P_p$ 's starting and end vertices have degree = 1
- the  $p - 2$  internal path vertices have degree  $\geq 2$
- the  $n_i + n_{i+1} - p$  remaining non-path vertices have degree = 1.

Note that as soon as a longest path  $P_p$  is found all remaining vertices not belonging to  $P_p$  are called non-path vertices.

*Proof.* The starting and end vertices of  $P_p$  have to have degree = 1, otherwise  $P_p$  would not be a longest induced path. The internal path vertices have to have degree  $\geq 2$  since only the starting and end vertices have degree = 1 and  $|\mathcal{C}_{G'}| = 1$  so no further component is possible. For the same reason the remaining

non-path vertices cannot have degree = 0. Assume there is a non-path vertex  $v$  with  $\deg(v) \geq 2$  which is adjacent to the path vertex  $x$ . Then there would be a MRNP tree pattern with  $x$  being the root vertex. So  $G'$  would no longer be radial planar, a contradiction.  $\square$

With this in mind we are able to count the edges of  $\mathcal{C}_1 = G'$ . There are  $p - 1$  path edges and  $n_i + n_{i+1} - p$  branches from the remaining non-path vertices. So the number of edges is exactly:

$$|E_1| = (p - 1) + (n_i + n_{i+1} - p) = n_i + n_{i+1} - 1. \quad (7)$$

**Case  $|\mathcal{C}_{G'}| \geq 2$  :** Compared to the cycle case connected components with more than one vertex are possible since path  $P_p$  leaves room for more edges from level  $l_i$  to level  $l_{i+1}$  in a component other than the one  $P_p$  belongs to. By assumption  $G'$  is a forest so each component in  $G'$  must have a longest induced path. Therefore the edge condition 7 has to apply to every single component in  $G'$ . We have  $|\mathcal{C}_{G'}|$  components. Since no cycles are allowed, we have edge maximal trees. So the **Edge Condition** is formed by adding the edges of each component:

$$|E'| = n_i + n_{i+1} - |\mathcal{C}_{G'}|. \quad (8)$$

Now we are able to phrase the exact number of edges in  $G'$ .

**Theorem 4.4.** *Let  $G$  be a graph with the induced subgraph  $G'$  between level  $l_i$  with vertices  $V_i$  and level  $l_{i+1}$  with vertices  $V_{i+1}$ . Let  $G'$  be a forest and  $P_p$  be a longest induced path of length  $p - 1$  in each component.  $G'$  is radial planar if and only if the Degree Condition for a path holds for each connected component of  $G'$  and the Edge Condition holds for  $G'$ .*

*Proof.* ' $\Rightarrow$ ' By assumption  $G'$  is radial planar and has no cycle so every connected component builds a tree. Hence the Edge Condition applies. The Degree Condition holds according to Lemma 4.3. If a component has no edges,  $K_1$ , the Degree Condition is not defined. But there are no edges to count so the component is subtracted in the Edge Condition.

' $\Leftarrow$ ' Assume the Degree and Edge Condition hold. Each component consists of either a  $K_1$  or has a longest induced path. Because of the Degree Condition we are able to place non-path vertices in between chords of the path vertex they are adjacent to. We can do so for every component since two components without a cycle cannot cross. Components consisting of a  $K_1$  are not relevant for the embedding. They cannot cause any crossing.  $\square$



**Corollary 4.5.** *If graph  $G$  is radial planar then the number of edges is exactly:*

$$|E| = \sum_{i=1}^{k-1} |E'_i|$$

where  $G'_i = (V'_i, E'_i)$  with  $V'_i = V_i \cup V_{i+1}$  is the induced subgraph between level  $l_i$  and level  $l_{i+1}$ .

$$|E'_i| = \begin{cases} n_i + n_{i+1} - |K_1|, & \text{if } G'_i \text{ contains a cycle} \\ n_i + n_{i+1} - |\mathcal{C}_{G'_i}|, & \text{if } G'_i \text{ is a forest.} \end{cases}$$

So counting the number of edges in every level span of size one is one way of preprocessing the graph. If a graph  $G$  does not meet Corollary 4.5 it cannot be radial planar.

The upper bound (4) can also be shown by using Corollary 4.5. Let  $|\tilde{K}_1| := \sum_{i=1}^{k-1} |K_1^i|$  where  $|K_1^i|$  is the amount of  $K_1$  in subgraph  $G'_i$ , if  $G'_i$  contains a cycle and is defined as in Corollary 4.5. In the same manner let  $|\tilde{\mathcal{C}}_G| := \sum_{i=1}^{k-1} |\mathcal{C}_{G'_i}|$ , if  $G'_i$  is a forest. So the number of edges in a radial planar graph  $G$  can be reformulated as follows

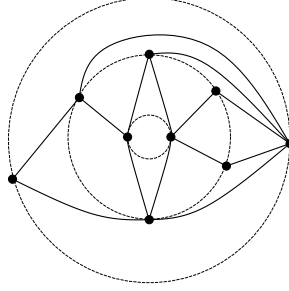
$$\begin{aligned} |E| &= \sum_{i=1}^{k-1} |E'_i| = n_1 + n_k + 2(n_2 + n_3 + \cdots + n_{k-1}) - |\tilde{K}_1| - |\tilde{\mathcal{C}}_G| \\ &= 2n - (n_1 + n_k + |\tilde{K}_1| + |\tilde{\mathcal{C}}_G|). \end{aligned} \tag{9}$$

Our aim is to prove inequality (4). Hence we are interested in the subtrahend of equation (9). We do know that  $n_1, n_k \geq 1$ , otherwise we would not have a  $k$  level graph.

**Case  $|\tilde{\mathcal{C}}_G| = 0$  :** There is no subgraph  $G'_i$  which contains a tree and no cycle. So there is a cycle of length at least four in every subgraph  $G'_i$ . Thus there have to be at least two vertices on every level, and therefore  $n_1, n_k \geq 2$ . So  $|E| \leq 2n - 4$  holds.

**Case  $|\tilde{\mathcal{C}}_G| = 1$  :** There is exactly one subgraph  $G'_i$  with one component which is not a cycle and all other subgraphs do contain a cycle. Suppose  $k = 2$ , so  $G$  is a 2 level graph. By assumption  $|V| \geq 3$ , hence  $n_1 \geq 2$  or  $n_2 \geq 2$  and (4) holds. Suppose  $k \geq 3$ , apart from one single subgraph all other subgraphs have to have a cycle since  $|\tilde{\mathcal{C}}_G| = 1$ . So all but one level have  $n_i \geq 2$ . Once again  $n_1 \geq 2$  or  $n_k \geq 2$  and (4) holds.

**Case  $|\tilde{\mathcal{C}}_G| \geq 2$  :**  $|E| \leq 2n - 4$  holds trivially.



**Figure 14:** An extremal graph with  $k = 3$ ,  $n = 9$  and  $m = 14$ .

All considerations have been done regardlessly of  $|\tilde{K}_1|$ , especially for  $|\tilde{K}_1| = 0$ . So we have been able to prove inequality (4) with the help of Corollary 4.5.

**Corollary 4.6.** *If graph  $G$  is radial planar then the number of edges is bounded.*

$$|E| \leq 2n - 4.$$

## 4.2 Extremal Graphs

We now have two tools to estimate and name respectively the amount of edges in a radial planar graph, namely Corollary 4.5 and 4.6. This enables us to analyze the radial planar graphs with a maximal amount of edges, saying the *extremal graphs*. Using equation (9) we have the equation for radial planar extremal graphs

$$n_1 + n_k + |\tilde{K}_1| + |\tilde{\mathcal{C}}_G| = 4. \quad (10)$$

Hence the extremal graphs are defined by the summands of equation (10). By assumption  $n_1, n_k \geq 1$  holds. The same case-by-case analysis as for Corollary 4.6 can be performed. Given a value for one summand the other summands have to be assigned regarding the fact that they all have to sum up to four.

**Case  $|\tilde{\mathcal{C}}_G| = 0$  :** There is no subgraph  $G'_i$  which consists of a forest. Hence there has to be a cycle of length at least four in every subgraph  $G'_i$  and at least two vertices on every level. Thus  $n_1 = n_k = 2$  and  $|\tilde{K}_1| = 0$  follows. See Figure 14 for an example.

**Case  $|\tilde{\mathcal{C}}_G| = 1$  :** There is exactly one subgraph  $G'_i$  with exactly one tree and no cycle. All other subgraphs have to contain a cycle. Remember,  $n_1, n_k \geq 1$ . So either  $|\tilde{K}_1| = 0$  or  $|\tilde{K}_1| = 1$  holds. In the latter case  $n_1 = n_k = 1$  would follow

which is not possible as we will see. Assume  $n_1 = n_k = 1$ . Then  $G'_1$  as well as  $G'_{k-1}$  exists of exactly one tree and no cycle, a contradiction to  $|\tilde{\mathcal{C}}_G| = 1$ .

If  $|\tilde{K}_1| = 0$  then  $n_1 = 1$  and  $n_k = 2$  or vice versa. Without loss of generality suppose  $n_1 = 1$  and  $n_k = 2$ . Thus  $G'_1$  is the only subgraph with a tree component. All other subgraphs  $G'_i$  consists of a cycle and possible branches. In particular  $G'_{k-1}$  contains a  $C_4$  because of  $n_k = 2$ . All remaining inner subgraphs  $G'_i$ ,  $i = 2, \dots, k-2$ , are build the same way as in the case of  $|\tilde{\mathcal{C}}_G| = 0$  since the difference lies in the values of  $|\tilde{\mathcal{C}}_G|$ ,  $|\tilde{K}_1|$ ,  $n_1$  and  $n_k$  only.

**Case  $|\tilde{\mathcal{C}}_G| = 2$ :** There is either one subgraph  $G'_i$  with exactly two components or two subgraphs with exactly one tree each. All other subgraphs must have a cycle. Nevertheless,  $|\tilde{K}_1| = 0$  since  $n_1, n_k \geq 1$ , to be specific  $n_1 = n_k = 1$  in this case. Therefore  $G_1$  and  $G_{k-1}$  have to exist of exactly one tree component each. Note that  $G_1$  and  $G_{k-1}$  only differ in their root vertex if  $k = 3$ . The same arguments as before can be applied to describe the subgraphs  $G'_i$ ,  $i = 2, \dots, k-2$ , for  $k \geq 4$ .

**Observation 4.7.** *A radial planar graph  $G$  with  $|\tilde{K}_1| \neq 0$  cannot be extremal.*

**Definition 4.8.** *Let  $G$  be a radial planar graph and  $G'_i = (V'_i, E'_i)$  with  $V'_i = V_i \cup V_{i+1}$  the induced subgraph between level  $l_i$  and level  $l_{i+1}$ .*

*Let  $|\tilde{K}_1| := \sum_{i=1}^{k-1} |K_1^i|$  where  $|K_1^i|$  is the amount of  $K_1$  in subgraph  $G'_i$ , if  $G'_i$  contains a cycle. In the same manner let  $|\tilde{\mathcal{C}}_G| := \sum_{i=1}^{k-1} |\mathcal{C}_{G'_i}|$ , if  $G'_i$  is a forest.*

*Let  $\mathcal{H}$  be the class of radial planar graphs which fulfill the equation*

$$n_1 + n_k + |\tilde{K}_1| + |\tilde{\mathcal{C}}_G| = 4 .$$

**Corollary 4.9.** *Let  $G = (V, E)$  be a radial planar graph with  $G \notin \mathcal{H}$  then  $|E| \leq 2n - 5$ .*

The class of graphs that fulfill the tighter bound  $|E| = 2n - 5$  can be achieved from the class  $\mathcal{H}$  by deleting an edge and possibly rearranging the vertices since three vertices on one of the outer level might be possible.

## 5 Conclusion

We have followed the idea of characterizing classes of graphs by forbidden subgraphs. The most common characterization for planar graphs in terms of minors comes from Kuratowski. By presenting MRNP patterns such a characterization has been made for hierarchical radial planar graphs as well in this work.

Radii can be pictured as circles on a sphere with a shared center, projected in

the plane we achieve the common representation of radial graphs. One extension of radii could be circles on a torus such that they all have the torus' axes as shared center. We then gain a new possibility to embed edges which has not been possible in the radial case. So MRNP patterns such as the loop augmented cycle can now be embedded toroidal. To look for a characterization of toroidal planar graphs is an interesting and unsolved problem.

We were also able to name a new upper bound for the number of edges of radial planar graphs. This can be used as a test for radial planarity in advance of an efficient algorithm to identify MRNP subgraphs. Developing the just mention algorithm is still an open question.

## References

- [1] G. Di Battista, E. Nardelli, Hierarchies and planarity theory, IEEE Trans. Systems Man Cybernet. 18 (6) (1988) 1035-1046.
- [2] R. Diestel, Graph Theory, Springer, (2010).
- [3] N. Hartsfield, G. Ringel, Pearls in Graph Theory, Dover Publications, INC., (2003).
- [4] P. Healy, A. Kuusik, S. Leipert, A characterization of level planar graphs, Discrete Math. 280 (2004) 51-63.
- [5] M. Jünger, S. Leipert, P. Mutzel, Level Planarity Testing in Linear Time, Report No. 99.369, Institut für Informatik, Universität zu Köln, (1999).
- [6] T. Nishizeki, Md. Saidur Rahman, Planar Graph Drawing, World Scientific Publishing Co. Pte. Ltd., Lecture Notes Series on Computing-Vol.12, (2004) .
- [7] H. Purchase, Graph Drawing, 5th International Symposium, 1997 Proceedings, 248-261.
- [8] B. Randerath, E. Speckenmeyer, E. Boros, P. Hammer, A. Kogan, K. Makino, B. Simeone, O. Cepeck, A Satisfiability Formulation of Problems on Level Graphs, Electronic Notes in Discrete Mathematics 9 (2001) 269-277.